ANCHOR MAPS AND STABLE MODULES IN DEPTH TWO

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ABSTRACT. An algebra extension $A\,|\,B$ is right depth two if its tensor-square $A \otimes_B A$ is in the Dress category $\mathbf{Add}\,_A A_B$. We consider necessary conditions for right, similarly left, D2 extensions in terms of partial A-invariance of two-sided ideals in A contracted to the centralizer. Finite dimensional algebras extending central simple algebras are shown to be depth two. Following P. Xu, left and right bialgebroids over a base algebra R may be defined in terms of anchor maps, or representations on R. The anchor maps for the bialgebroids $S = \mathrm{End}\,_B A_B$ and $T = \mathrm{End}\,_A A \otimes_B A_A$ over the centralizer $R = C_A(B)$ are the modules $_S R$ and R_T studied in [12, 16, 9], which provide information about the bialgebroids and the extension [10]. The anchor maps for the Hopf algebroids in [19, 11] reverse the order of right multiplication and action by a Hopf algebra element, and lift to the isomorphism in [22]. We sketch a theory of stable A-modules and their endomorphism rings and generalize the smash product decomposition in [8, Prop. 1.1] to any A-module. We observe that Schneider's coGalois theory in [23] provides examples of codepth two, such as the quotient epimorphism of a finite dimensional normal Hopf subalgebra. A homomorphism of finite dimensional coalgebras is codepth two if and only if its dual homomorphism of algebras is depth two.

1. INTRODUCTION AND PRELIMINARIES

Anchor maps for bialgebroids are defined algebraically by Ping Xu [27] based on quantization of certain triangular Lie bialgebroids. From this point of view, a classical cocommutative bialgebroid such as the univ. env. algebra of a Lie algebroid has an anchor map extending the anchor map of a Lie algebroid, which is a bundle map from a real vector bundle E over a smooth manifold X into the tangent bundle $T(X)$ [26]. For example, $T(X)$ itself has bialgebroid with total algebra $\mathcal{D}(X)$, the algebra of diffential operators on X, base algebra $C^\infty(X)$ with anchor map $\mu : \mathcal{D}(X) \to \mathrm{End}\, C^\infty(X)$ the usual action of differential operators on smooth functions. The anchor map of a Lie algebroid E pulls back from $T(X)$ a considerable amount of differential geometry such as Lie bracket, connection and De Rham cohomology [26]. Analogously, the anchor map of a bialgeboid encodes in the unit module of the associated monoidal category information about the bialgebroid: we will provide some evidence for this, mentioned in the abstract and treated in some detail in section 3.

Depth two extensions arise very naturally from a subalgebra pair $B \subseteq A$ satisfying a certain projectivity condition on the tensor-square. For example, a Galois extension $A\,|\,B$ over a projective R-bialgebroid H is depth two since $A \otimes_B A \cong A \otimes_R H$ as A-B-bimodules, where B and R are commuting subalgebras within A [13]. This in fact characterizes depth two extension or at least its endomorphism ring extension [14]. A simpler example occurs if B is in the center of A, then $B \subseteq A$ is depth

two if A is finite projective as a B-module. If we think of a central simple algebra B as a "noncommutative point" we might expect that any finite dimensional algebra A extending B be depth two. We provide a rigourous proof of this and a somewhat more general fact in section 2.

For a quantum subalgebra such as sub-group algebra, sub-Hopf algebras, twisted and skew variants of these, the notion of depth two is closely related to, perhaps characterizes, the notion of normal subobject. In the Clifford theory of decomposition of induced modules from a normal subgroup $H \triangleleft G$, certain modules are stable over the normal subgroup, i.e., are isomorphic to all conjugate modules. The endomorphism ring of the induced module of a stable module has essentially the structure of a twisted group algebra over the quotient algebra G/H by a result of Conlon [5, 11C]. Schneider [24] extends this and other classical results by Clifford, Green and Blattner, unified within the induced representation theory of Hopf-Galois extensions. In section 4 we take the point of view that f.g. Hopf-Galois extensions (such as the finite strongly group-graded algebras or graded Clifford systems) are depth two and certain constructions in [24] such as stable modules will sensibly generalize. The endomorphism ring of an induced, restricted left A-module M is for example a smash product of the depth two right bialgebroid T acting on $\operatorname{End}_B M$ over the centralizer $R = A^B$ (Theorem 4.1).

Finally by way of introduction, the question of whether a depth two Hopf subalgebra is normal in a finite dimensional Hopf algebra leads to a notion of codepth two homomorphism of coalgebras $C \to D$ in the author's paper [15], since Hopf algebra homomorphisms are also homomorphisms of coalgebras. As we show in section 5, a homomorphism of finite dimensional coalgebras is codepth two if and only if its dual homomorphism of algebras is depth two. Schneider's coGalois theory in [23] provides an answer to a question of whether $H \to H/HK^+$ is codepth two for normal Hopf subalgebras, the answer being supplied in section 5.

1.1. **Preliminaries.** We let $\mathbf{Add}\, M$ denote the Dress category of a module M_C over a ring C, consisting of all C-modules isomorphic to direct summands of finite direct sums $M \oplus \cdots \oplus M$, and all module homomorphisms between these. We let $\mathbf{FGP}\, C$ denote the category of finitely generated, projective right C-modules and all module homomorphisms between these. Recall that $\mathbf{Add}\, M$ is equivalent to the category $\mathbf{FGP}\operatorname{End} M_C$ via the functor $X_C \mapsto \operatorname{Hom}_C(M, X)$, where $\operatorname{Hom}_C(M, X)$ is a right module over $E = \operatorname{End} M_C$ via ordinary composition. (Its inverse functor is given by $- \otimes_E M$ where $_E M_C$ is the natural bimodule.)

Let $A \,|\, B$ be a unital associative algebra extension, such as subring $B \subseteq A$ with $1_B = 1_A$ or a unital ring homomorphism $B \to A$. Let k denote the ground ring, a field in the later sections 4 and 5. Note that the natural B-A-bimodule A is in $\mathbf{Add}\, A \otimes_B A$, since the multiplication mapping $A \otimes_B A \to A$ is a B-A-split epi. The same is true of $_A A_B \in \mathbf{Add}\, A \otimes_B A$.

The converse condition defines the notion of depth two. The extension $A \,|\, B$ is depth two (D2) if $A \otimes_B A \in \mathbf{Add}\, A$ as natural A-B-bimodules (right D2) and B-A-bimodules (left D2).

Note that $\operatorname{End}_B A_A \cong C_A(B)$, the centralizer of the extension, which we denote by $R = C_A(B)$, via $r \mapsto \lambda_r$, left multiplication of A by $r \in R$. There is then a category equivalence $\mathbf{Add}\, A \cong \mathbf{FGP}\, R$; in particular, $\operatorname{Hom}\,(_B A_A, {}_B A \otimes_B A_A) := T$

is finitely generated projective as a right R-module. Note that

$$T \cong (A \otimes_B A)^B \cong \operatorname{End}{}_A A \otimes_B A_A \tag{1}$$

via $f \mapsto f(1)$, which we take as an identification of T with the B-commutator of $A \otimes_B A$, and $t \mapsto (a \otimes_B a' \mapsto at^1 \otimes_B t^2 a')$ respectively, where we denote $t = t^1 \otimes_B t^2 \in T$ (notationally suppressing any summation over simple tensors). The last isomorphism induces the ring structure on $T = (A \otimes_B A)^B$ given by

$$uv = v^1 u^1 \otimes_B u^2 v^2, \quad 1_T = 1_A \otimes_B 1_A. \tag{2}$$

Given $\mathbf{Add}\, M_C = \mathbf{Add}\, N_C$, it follows that that $\operatorname{End} M_C$ and $\operatorname{End} N_C$ are Morita equivalent (Hirata, 1968). In particular, R and $\operatorname{End}{}_B A \otimes_B A_A$ are Morita equivalent. The inverse equivalence of course comes from

$$\operatorname{Hom}\left({}_B A \otimes_B A_A,\, {}_B A_A\right) \cong \operatorname{End}{}_B A_B := S$$

via $f \mapsto f(- \otimes_B 1_A)$. Thus, a left D2 extension $A \,|\, B$ has f.g. projective R-module structures ${}_R S$ and T_R on the rings S and T. Similarly, ${}_R T$ and S_R are finite projective R-modules for a right D2 extension $A \,|\, B$.

In case $A \,|\, B$ is additionally a Frobenius extension, we have the algebraic structure of the Jones tower of a type II_1 subfactor:

$$B \to A \hookrightarrow A_1 \hookrightarrow A_2$$

where $A_1 = \operatorname{End} A_B \cong A \otimes_B A$ and $A_2 = \operatorname{End}(A_1)_A \cong \operatorname{End} A \otimes_B A_A$. In the case of depth two, the relative commutators $R = C_A(B)$ and $C_{A_2}(B) = \operatorname{End}{}_B A \otimes_B A_A$ are Morita equivalent with context bimodules $C_{A_1}(B) \cong \operatorname{End}{}_B A_B$ and $C_{A_2}(A) \cong \operatorname{End}{}_A A \otimes_B A_A$, i.e. S and T. Thus the notion of depth two algebra extension recovers classical depth two for subfactors ([18, 16] for further details). Below we examine a pairing between S and T that becomes [18, 8.9] the Szymański nondegenerate pairing of $C_{A_1}(B)$ and $C_{A_2}(A)$ in [17, (14)], which transfers the algebra structure of one centralizer to a coalgebra structure on the other when R is trivial.

The following coordinates for left and right depth two extensions are useful for concrete computation. Given a left D2 extension $A \,|\, B$, we have a split epi $A^n \to A \otimes_B A$ and thus a left D2 quasibase $\beta_i \in S, t_i \in T$ satisfying in $A \otimes_B A$:

$$x \otimes y = \sum_{i=1}^{n} t_i \beta_i(x) y. \tag{3}$$

Assuming a left D2 quasibase for an extension is equivalent to our defining condition above: define a split epi $\pi : A^n \to A \otimes_B A$ by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} t_i a_i$ with section $\sigma : x \otimes y \mapsto (\beta_1(x)y, \ldots, \beta_n(x)y)$. (The corresponding B-A-endomorphism $\sigma \circ \pi : A^n \to A^n$ is given by the idempotent matrix $\underline{\mathbf{r}} = (r_{ij})$ in $M_n(R)$, where $r_{ij} = \sum_{(t_j)} \beta_i(t_j^1) t_j^2$.)

Similarly, given a right D2 extension $A \,|\, B$, there is a right D2 quasibase: $\exists \gamma_j \in S, u_j \in T$ such that

$$x \otimes y = \sum_{j=1}^{m} x \gamma_j(y) u_j. \tag{4}$$

We will fix this notation for left and right D2 quasibases throughout this paper.

1.2. **Mapping viewpoint on R-valued pairings between S and T.** Given a algebra extension $A\,|\,B$, recall the pairings of $S = \mathrm{End}\,_B A_B$ and $T = (A \otimes_B A)^B$ given by $\langle \alpha, t \rangle = \alpha(t^1)t^2$ and $[\beta, u] = u^1\beta(u^2)$ both with values in $R = C_A(B)$. If $A\,|\,B$ is D2, these are nondegenerate pairings inducing $_R S \overset{\cong}{\longrightarrow} {_R}\mathrm{Hom}\,(T_R, R_R)$ and $S_R \overset{\cong}{\longrightarrow} \mathrm{Hom}\,(_R T, {_R}R)_R$, respectively [18].

Now note $\mathrm{End}\,_B A_B \overset{\cong}{\longrightarrow} \mathrm{Hom}\,(_A A \otimes_B A \otimes_B A_A, {_A}A_A)$ via $\alpha \mapsto \mu^2 \circ (\mathrm{id}_A \otimes \alpha \otimes \mathrm{id}_A)$, with inverse $F \mapsto F(1 \otimes - \otimes 1)$. Suppose F and $\alpha \in S$ are images of one another under these mappings; suppose $G \in \mathrm{End}\,_A A \otimes_B A_A$ and $t \in T$ are images of one another under the mappings in eq. (1) above. There are two obvious B-linear mappings $A \otimes_B A \rightrightarrows A \otimes_B A \otimes_B A$ given by $x \otimes y \mapsto 1 \otimes x \otimes y$ or $x \otimes y \otimes 1$. Then the two pairings $\langle \alpha, t \rangle$ and $[\alpha, t]$ are equal to the two images of $1_A \otimes_B 1_A$ under composition of the following mappings,

$$1 \otimes 1 \in A \otimes_B A \xrightarrow{G} A \otimes_B A \rightrightarrows A \otimes_B A \otimes_B A \xrightarrow{F} A \tag{5}$$

since $\mathrm{Hom}\,(_B A \otimes_B A_B, {_B}A_B) \longrightarrow R$ via $H \mapsto H(1 \otimes 1)$.

2. Left and right depth two

In [9] the author observed that the centralizer of a depth two extension is a normal subring in Rieffel's sense. Related to this, we point out a necessary condition for an algebra extension to be left D2, resp. right D2.

Proposition 2.1. *Let $A\,|\,B$ be an algebra extension with centralizer $R = C_A(B)$. If $A\,|\,B$ is left D2, then for each two-sided ideal $I \subseteq A$, the ideal contracted to R satisfies left partial A-invariance:*

$$A(I \cap R) \subseteq (I \cap R)A \tag{6}$$

If $A\,|\,B$ is right D2, then for each two-sided ideal $I \subseteq A$, the ideal contracted to R satisfies right partial A-invariance:

$$(I \cap R)A \subseteq A(I \cap R) \tag{7}$$

Proof. Given $s \in I \cap R$ and $a \in A$ and a left D2 quasibase, we note from eq. (3) that

$$as = \sum_i t_i^1 s t_i^2 \beta_i(a) \in (I \cap R)A$$

since $t_i \in (A \otimes_B A)^B$.

Similarly, from a right D2 quasibase we obtain

$$sa = \sum_j \gamma_j(a) u_j^1 s u_j^2 \in A(I \cap R),$$

whence the second of the two set inclusions. $\square$

The conditions of left and right partial A-invariance may be compared with invariance under the left and right adjoint actions (ad_ℓ- and ad_r-invariance) of a Hopf subalgebra: the identities $a_{(1)} \otimes \tau(a_{(2)})a_{(3)} = a \otimes 1$ and $a_{(1)}\tau(a_{(2)}) \otimes a_{(3)} = 1 \otimes a$, for an element a in a Hopf algebra with antipode τ, are comparable to the eqs. (3) and (4) (cf. [9, 21]). With this criteria for one-sided D2 extensions, it is often easy to identify certain extensions as not being D2.

Example 2.2. Suppose U and V are rings, while $_U M_V$ and $_V N_U$ are bimodules. Form a special case of the generalized matrix ring $A = \begin{pmatrix} U & M \\ N & V \end{pmatrix}$, a fiber product of upper and lower matrix algebras with multiplication given by

$$\begin{pmatrix} u & m \\ n & v \end{pmatrix} \begin{pmatrix} u' & m' \\ n' & v' \end{pmatrix} = \begin{pmatrix} uu' & um' + mv' \\ nu' + vn' & vv' \end{pmatrix}.$$

Let the subring $B = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$. Then the centralizer $R = \begin{pmatrix} Z(U) & 0 \\ 0 & Z(V) \end{pmatrix}$ where $Z(U)$ and $Z(V)$ are centers of U and V. Consider the two-sided ideal $I = \begin{pmatrix} U & M \\ N & 0 \end{pmatrix}$ in A. We note that $R \cap I = \begin{pmatrix} Z(U) & 0 \\ 0 & 0 \end{pmatrix}$, $A(R \cap I) = \begin{pmatrix} U & 0 \\ N & 0 \end{pmatrix}$, and $(R \cap I)A = \begin{pmatrix} U & M \\ 0 & 0 \end{pmatrix}$. If $M \neq 0$ and $N \neq 0$, then $A(R \cap I) \not\subset (R \cap I)A$ and $(R \cap I)A \not\subset A(R \cap I)$, so $A \,|\, B$ is neither left nor right D2 (respectively).

If $N = 0$ and $M \neq 0$, A is an upper triangular matrix and B the "diagonal" subring. Choose instead the two-sided ideal $I = \begin{pmatrix} U & M \\ 0 & 0 \end{pmatrix}$, show that $A(R \cap I) \not\subset (R \cap I)A$, so that $A \,|\, B$ is not right D2. It is not left D2 by using instead the ideal $J = \begin{pmatrix} 0 & M \\ 0 & V \end{pmatrix}$.

It is an open problem if there exists a left D2 algebra extension which is not right D2 (or the reverse if we pass to opposite algebras). The test above for the one-sided D2 property might be helpful in finding such an algebra extension, if a certain extension showed signs of being depth two, with centralizer intermediate in size and the over-algebra having a sufficiently rich ideal structure.

We now turn to an example of depth two extension.

Theorem 2.3. *Suppose B is an Azumaya k-algebra and A is a fin. gen. projective k-algebra containing B as a subalgebra. Then $A \supseteq B$ is a depth two extension.*

Proof. Since B is Azumaya, it is known (e.g. [7, p. 46]) that there are Casimir elements $e_i \in (B \otimes_k B)^B$ and elements $b_i \in B$ such that

$$1 \otimes_k 1 = \sum_{i=1}^{N} e_i b_i. \tag{8}$$

Then for all $b \in B, a \in A$

$$b \otimes_k a = \sum_i e_i b b_i a = \sum_i f_i(g_i(b \otimes_k a))$$

where $g_i : B \otimes_k A \to A$ is defined by $g_i(b \otimes_k a) = bb_i a$ and $f_i : A \to B \otimes_k A$ is defined by $f_i(x) = e_i x$ for each $i = 1, \ldots, N$. It follows that

$$_B B \otimes_k A_A \oplus * \cong {}_B A_A^N.$$

Since B is separable as a k-algebra, B^e is a semisimple extension of k. (E.g. if k is a field, B is semisimple and so is $B^e = B \otimes_k B^{\mathrm{op}}$.) Then $_B A_B$ is k-relative projective, and projective since A is projective over k. Then $_B A_B \oplus * \cong {}_B B \otimes_k B_B^M$. Tensoring by $- \otimes_B A_A$, we obtain

$$_B A \otimes_B A_A \oplus * \cong {}_B B \otimes_k A_A^M.$$

Combining the two displayed isomorphisms, we obtain $A \otimes_B A \oplus * \cong A^{NM}$ as the natural B-A-bimodules, whence $A \mid B$ is left D2. It is similarly argued that $A \mid B$ is right D2. □

If $a_k \in A$ and $G_k \in \mathrm{Hom}\,(A, B^e)$ $(k = 1, \ldots, M)$ denote a finite projective basis for A as a right B^e-module, the proof above converts to the left D2 quasibase for $A \mid B$: $(x, y \in A)$

$$x \otimes_B y = \sum_{i,k} e_i^1 a_k \otimes_B e_i^2 (b_i \cdot G_k(x)) y \tag{9}$$

where $\beta_{ik}(x) = b_i \cdot G_k(x)$ are in fact B-valued endomorphisms in $\mathrm{End}\,{}_B A_B$ and $t_{ik} = e_i^1 a_k \otimes_B e_i^2$ are in $R \otimes 1 \subseteq (A \otimes_B A)^B$.

We will study elsewhere the more general setting of composite extensions $A \mid B \mid C$, where $A \mid B$, $A \mid C$ are D2 extensions and $B \mid C$ is H-separable, the total rings underlying the bialgebroids $T_{A|B} = (A \otimes_B A)^B$ and $T_{A|C} = (A \otimes_C A)^C$ are Morita equivalent in an interesting way. The Morita context bimodules are $P = (A \otimes_B A)^C$ and $Q = (A \otimes_C A)^B$, admitting a calculus extending that in [18] and the centralizers A^B and A^C are functorial images of one another under tensoring by P and Q.

3. Bialgebroids in terms of anchor maps

A bialgebroid H over a base ring R is usually defined as an R^e-ring, R-coring with grouplike element 1_H and an augmentation ring $\varepsilon : H_H \to R_H$ with multiplicative coproduct, a definition that stays closest to the usual definition of a Hopf algebra as an augmented algebra, coalgebra with homomorphic coproduct. However, there is a slightly different, equivalent way of defining a bialgebroid which comes from the theory of Lie algebroids, their universal enveloping algebras (which are cocommutative Hopf algebroids) and quantized variants of these. This is Ping Xu's definition [27] of a left R-bialgebroid in terms of an anchor mapping, instead of a counit, an anchor map being a representation of H on R yielding the unit module in the tensor category of H-modules. In this section we give this definition for right bialgebroids and show some useful aspects of the anchor mapping.

The definition of a right bialgebroid with total algebra H and base algebra R in terms of an anchor mapping is the following (cf. [1, 20, 27] for corresponding definition of left bialgebroid):

(1) H and R are unital, associative k-algebras,
(2) there are commuting algebra homomorphisms $R \xrightarrow{\sigma} H \xleftarrow{\tau} R^{\mathrm{op}}$ (called *source* and *target*, respectively) where for each $r, s \in R$ we have $\sigma(r)\tau(s) = \tau(s)\sigma(r)$,
(3) fix the R-R-bimodule ${}_R H_R$ given by

$$r \cdot h \cdot s = h\sigma(s)\tau(r) \quad (h \in H)$$

(4) H has R-R-bilinear *comultiplication* $\Delta : H \to H \otimes_R H$ given by notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ where $\Delta(1_H) = 1_H \otimes_R 1_H$.
(5) multiplicativity of Δ with technical pre-condition: for all $h, g \in H$, $r \in R$,

$$h_{(1)} \otimes_R \tau(r)h_{(2)} = \sigma(r)h_{(1)} \otimes_R h_{(2)} \tag{10}$$

$$\Delta(hg) = \Delta(h)\Delta(g) \tag{11}$$

(6) an (anchor) map $\mu : H \to \operatorname{End}_k R$, an R-R-bimodule morphism w.r.t. the R-R-bimodule $\operatorname{End} R$ given by $r \cdot f \cdot s = rf(-)s$, an anti-homomorphism and right action of H on R (whence we write $\mu(h)(r) := r \triangleleft h$) satisfying:

(7) $h_{(1)}\sigma(r \triangleleft h_{(2)}) = \sigma(r)h$

(8) $h_{(2)}\tau(r \triangleleft h_{(1)}) = \tau(r)h$.

The existence of an anchor map is equivalent to the existence of a counit $\varepsilon : H \to R$ (cf. [1], (H, Δ, ε) becomes an R-coring). For example, set $\varepsilon(h) = 1_R \triangleleft h$, which is R-bilinear since μ is, and note that $\varepsilon(1_H) = 1_R$, $\varepsilon(h_{(1)}) \cdot h_{(2)} = h_{(2)}\tau(1 \triangleleft h_{(1)}) = h$ and

$$\varepsilon(\sigma(\varepsilon(h))g) = 1 \triangleleft \sigma(\varepsilon(h))g = (1 \triangleleft h) \triangleleft g = \varepsilon(hg).$$

Conversely, given the counit $\varepsilon : H \to R$ the anchor is computed from

$$\mu(h)(r) = \varepsilon(\sigma(r)h) = \varepsilon(\tau(r)h). \tag{12}$$

In the tensor category of right H-modules, the anchor mapping is the unit module structure on R, so that for each H-module V, $R \otimes_R V \cong V$ as well as $V \otimes_R R \cong V$ as H-modules. The anchor map may also be viewed as an arrow into the terminal object $(\operatorname{End} R)^{\operatorname{op}}$ in the category of R-bialgebroids.

Example 3.1. Let $A \,|\, B$ be a right D2 extension. We recall that $T = (A \otimes_B A)^B$ is a right bialgebroid over the centralizer $R = C_A(B)$ ([18, 5.2], two-sided depth two is not needed in the argument). The ring structure on T is given in eq. (2). Note that $\sigma(r) = 1 \otimes_B r$ and $\tau(s) = s \otimes_B 1$ define homomorphism and anti-homomorphism $R \to T$ that commute at all values and induce from the right the R-R-bimodule ${}_R T_R$ given by $s \cdot t \cdot r = st^1 \otimes_B t^2 r$. The comultiplication $\Delta : T \to T \otimes_R T$ given by $\Delta(t) = \sum_j (t^1 \otimes_B \gamma_j(t^2)) \otimes_R u_j$ and anchor mapping

$$\mu(t)(r) = r \triangleleft t = t^1 r t^2 \tag{13}$$

are R-bilinear and satisfy

$$\sigma(r)t = t^1 \otimes_B rt^2 = (t^1 \otimes_B \gamma_j(t^2))(1 \otimes_B u_j^1 r u_j^2) = t_{(1)}\sigma(r \triangleleft t_{(2)})$$

and similarly $\tau(r)t = t_{(2)}\tau(r \triangleleft t_{(1)})$. The counit is given by $\varepsilon(t) = \mu(t)(1) = t^1 t^2$. We note that the representation μ is the module algebra R_T and studied in [16, 9] as a generalized Miyashta-Ulbrich action.

The corresponding anchor mapping based definition of left bialgebroid is the opposite of the definition above, given in detail in [1]. Again we are interested in the example coming from a depth two extension $A \,|\, B$. We recall the left bialgebroid structure on the endomorphism ring $S = \operatorname{End}{}_B A_B$ over R. The source and target mapping $R \to S \leftarrow R^{\operatorname{op}}$ are provided by the standard left and right multiplication mappings $\lambda_r : x \mapsto rx$ and $\rho_s : x \mapsto xs$ for $r, s \in R$. Of course, $\lambda_r \circ \rho_s = \rho_s \circ \lambda_r$ and ${}_R S_R$ is given by

$$r \cdot \alpha \cdot s = \lambda_r \circ \rho_s \circ \alpha = r\alpha(-)s$$

for $\alpha \in S$. The comultiplication $\Delta : S \to S \otimes_R S$ is given by

$$\Delta(\alpha) = \sum_j \gamma_j \otimes_R (\alpha \triangleleft u_j) \tag{14}$$

where $\alpha \triangleleft t = t^1 \alpha(t^2 -)$ is an action of T on S discussed in more detail in the next section. The counit $\varepsilon : S \to R$ given by $\varepsilon(\alpha) = \alpha(1)$ together with Δ provides the

R-R-bimodule S an R-coring structure. Then the anchor map $\mu : S \to \operatorname{End} R$ is given by

$$\mu(\alpha)(r) = \varepsilon(\alpha \circ \lambda_r)(1) = \alpha(r) \tag{15}$$

This gives R the structure of a left S-module algebra. The underlying module $_S R$ has been studied in [12, 9].

As shown in the next example, a comparison of anchor maps may lift to an isomorphism between the bialgebroids they represent.

Example 3.2. Suppose H is a Hopf algebra with antipode τ and A a left H-module algebra. Consider two left A-bialgebroid structures on the total space $A \otimes_k A \otimes_k H$ which have been studied recently. In [11] the left bialgebroid $A^e \bowtie H$ is defined (by considering the S construction for a (special depth two) pseudo-Galois extension) with multiplication given by

$$(a \otimes b \bowtie h)(c \otimes d \bowtie k) = a(h_{(1)} \rhd c) \otimes d(\tau(k_{(2)}) \rhd b) \bowtie h_{(2)} k_{(1)}, \tag{16}$$

See the paper [11] for the details; here, we will only need to know that the source $s_L : A \to A^e \bowtie H$ is given by $s_L(a) = a \otimes 1_A \bowtie 1_H$, and the counit by

$$\varepsilon(a \otimes b \bowtie h) = a(h \rhd b). \tag{17}$$

We compute the anchor map $\mu : A^e \bowtie H \to \operatorname{End} A$:

$$\mu(a \otimes b \bowtie h)(x) = \varepsilon((a \otimes b \bowtie h)(x \otimes 1_A \bowtie 1_H))$$

$$= \varepsilon(a(h_{(1)} \rhd x) \otimes b \bowtie h_{(2)}) = a(h \rhd xb)$$

whence $(a, b \in A, h \in H)$

$$\mu(a \otimes b \bowtie h) = \lambda_a \circ (h \rhd \cdot) \circ \rho_b. \tag{18}$$

In the papers [4, 19] the same data inputs into a left bialgebroid $A \odot H \odot A$ where multiplication is given by

$$(a \odot h \odot b)(c \odot k \odot d) = a(h_{(1)} \rhd c) \odot h_{(2)} k \odot (h_{(3)} \rhd d)b \tag{19}$$

source homomorphism by $\lambda(a) = a \odot 1_H \odot 1_A$ and counit by

$$\varepsilon(a \odot h \odot b) = \varepsilon_H(h)ab \tag{20}$$

where ε_H is of course the counit of the Hopf algeba H. The anchor for this bialgebroid is then $(a, b, x \in A, h \in H)$:

$$\mu(a \odot h \odot b)(x) = \varepsilon((a \odot h \odot b)(x \odot 1_H \odot 1_A) = \varepsilon(a(h_{(1)} \rhd x) \odot h_{(2)} \odot b) = a(h \rhd x)b$$

i.e.

$$\mu(a \odot h \odot b) = \lambda_a \circ \rho_b \circ (h \rhd \cdot) \tag{21}$$

Observe that

$$\lambda_a \circ (h \rhd \cdot) \circ \rho_b = \lambda_a \circ \rho_{h_{(2)} \rhd b} \circ (h_{(1)} \rhd \cdot),$$

which lifts to an isomorphism of bialgebroids $A^e \bowtie H \cong A \odot H \odot A$ given by

$$a \otimes b \bowtie h \longmapsto a \odot h_{(1)} \odot h_{(2)} \rhd b \tag{22}$$

given in [22] (with inverse, $a \odot h \odot b \mapsto a \otimes \tau(h_{(2)}) \rhd b \bowtie h_{(1)}$, a bialgebroid homomorphism commuting with source, target and counit maps and is an A-coring homomorphism [2]).

4. On stable modules and their endomorphism rings

Suppose $_A M$ is a left A-module. Let $\mathcal{E}$ denote its endomorphism ring as a module restricted to a B-module: $\mathcal{E} = \operatorname{End}_B M$. There is a right action of T on $\mathcal{E}$ given by $f \triangleleft t = t^1 f(t^2 -)$ for $f \in \mathcal{E}$. This is a measuring action and $\mathcal{E}$ is a right T-module algebra (as defined in [18, 2]), since

$$(f \triangleleft t_{(1)}) \circ (g \triangleleft t_{(2)}) = \sum_i t_i^1 f(t_i^2 \beta_i(t^1) g(t^2 -)) = (f \circ g) \triangleleft t.$$

The subring of invariants in $\mathcal{E}$ is $\operatorname{End}_A M$ since $\operatorname{End}_A M \subseteq \mathcal{E}^T$ is obvious, and $\phi \in \mathcal{E}^T$ satisfies for $m \in M, a \in A$:

$$\phi(am) = \sum_j \gamma_j(a)(\phi \triangleleft u_j)(m) = \sum_j \gamma_j(a)\varepsilon_T(u_j)\phi(m) = a\phi(m).$$

The next theorem shows that the endomorphism ring of the induced module is the smash product ring of the bialgebroid T with the endomorphism ring $\mathcal{E}$ (generalizing [8, 1.1, $M = A$]), the isomorphism $\Psi : T \ltimes \mathcal{E} \longrightarrow \operatorname{End}_A A \otimes_B M$ being given by

$$\Psi(t \otimes f)(a \otimes m) = at^1 \otimes_B t^2 f(m). \tag{23}$$

Theorem 4.1. *Let M be a left A-module and $\mathcal{E} = \operatorname{End}_B M$. If $A \,|\, B$ is left depth two, then there is a ring isomorphism $\Psi : T \ltimes \mathcal{E} \overset{\cong}{\longrightarrow} \operatorname{End}_A A \otimes_B M$.*

Proof. We let $\mu_M : A \otimes_B M \to M$ denote the multiplication mapping defined by $a \otimes m \mapsto am$. Letting $F \in \operatorname{End}_A A \otimes_B M$, define $\Phi : \operatorname{End}_A A \otimes_B M \to T \otimes_R \mathcal{E}$ by

$$\Phi(F) = \sum_j t_j \otimes_R \mu_M \circ (\beta_j \otimes_B \operatorname{id}_M)F(1_A \otimes -).$$

Note that $\Phi \circ \Psi = \operatorname{id}$, since for $t \otimes f \in T \otimes_R \mathcal{E}$,

$$\sum_j t_j \otimes_R \mu_M(\beta_j \otimes \operatorname{id}_M)(t^1 \otimes t^2 f(-)) = \sum_j t_j \beta_j(t^1) t^2 \otimes f = t \otimes f.$$

Next, given $F \in \operatorname{End}_A A \otimes_B M$, let $F^1(m) \otimes F^2(m) := F(1 \otimes m)$ noting that $F(a \otimes m) = aF^1(m) \otimes F^2(m)$. Observe that $\Psi\Phi = \operatorname{id}$ since

$$\Psi\Phi(F)(a \otimes m) = \sum_j at_j^1 \otimes t_j^2 \beta_j(F^1(m))F^2(m) = aF^1(m) \otimes F^2(m) = F(a \otimes m).$$

Thus Ψ is bijective linear mapping.

Verify that Ψ is a ring isomorphism, using $\Delta(u) = \sum_j t_j \otimes_R (\beta_j(u^1) \otimes_B u^2)$:

$$
\begin{aligned}
\Psi((t \ltimes f)(u \ltimes g))(a \otimes m) &= \Psi(tu_{(1)} \ltimes (f \triangleleft u_{(2)})g)(a \otimes m) \\
&= \sum_j at_j^1 t^1 \otimes t^2 t_j^2 \beta_j(u^1) f(u^2 g(m)) \\
&= au^1 t^1 \otimes t^2 f(u^2 g(m)) \\
&= \Psi(t \ltimes f) \circ \Psi(u \ltimes g)(a \otimes m). \quad \square
\end{aligned}
$$

With $M = A$, we obtain the isomorphism,

$$T \ltimes \operatorname{End}_B A \cong \operatorname{End}_A A \otimes_B A \tag{24}$$

Note that if $A \,|\, B$ is D2, the module $_A A \otimes_B A$ is finite projective and a generator since $\mu : A \otimes_B A \to A$ splits as a left A-module epi. Then $\operatorname{End}_A A \otimes_B A$ is Morita equivalent to A. If $A \,|\, B$ is a Frobenius extension, $\operatorname{End}_B A$ is a smash product of

A and S. In this case, eq. (24) may be viewed as a duality result for D2 Frobenius extensions (cf. [8], [21, ch. 9]).

The theory of stable B-modules has the intent to generalize a smash product result like the one above to a certain extent. We sketch the beginnings of such a project by extending the definition of stable modules and certain theorems in Schneider [24] to the bialgebroid-Galois extensions. (Recall from [13] that such Galois extensions are characterized by being D2 and balanced.)

Suppose that $A \mid B$ is a faithfully flat, balanced, depth two extension of algebras over a field k. Let R again be the centralizer $C_A(B)$, let T^{op} be the left bialgebroid $(A \otimes_B A)^B$ over R with the opposite multiplication of that in eq. (2) and identical to T as R-corings, and BR the smallest subalgebra in A containing B (or the image of B) and R. We recall that A is a left T^{op}-comodule algebra with coinvariants equal to B (by faithful flatness of the natural module $_B A$) [12]; in particular, A is a left T-comodule. Of course, T is a left T-comodule over itself (see [2] for comodules over corings).

Definition 4.2. *Suppose M is a left BR-module. We say that M is A-stable if*

$$A \otimes_B M \cong T \otimes_R M \tag{25}$$

by a left B-linear and left T-colinear isomorphism.

This definition is most useful when $B = BR$, e.g. a maximal commutative subalgebra of A or a trivial centralizer $k1_A$ (then our bialgebroids are bialgebras and Galois extensions are Hopf-Galois extensions). If $BR = A$ we are in the situation below, that all A-modules are A-stable. Recall our notation $t_i \in T$, $\beta_i \in S$ for a left D2 quasibasis.

Proposition 4.3. *Any left A-module M is A-stable via the isomorphism*

$$\Psi: \ A \otimes_B M \xrightarrow{\cong} T \otimes_R M, \ \ a \otimes_B m \longmapsto \sum_i t_i \otimes_R \beta_i(a)m \tag{26}$$

Proof. We note that Ψ is left B-linear since $\Psi(ba \otimes_B m) = \sum_i t_i \otimes_R \beta_i(ba)m = b\Psi(a \otimes_B m)$ since $\beta_i \in \mathrm{End}\,_B A_B$. Recall that A is a left T-comodule via $\rho_L(a) = \sum_i t_i \otimes_R \beta_i(a)$, and T has coproduct $\Delta(t) = \sum_i t_i \otimes_R (\beta(t^1) \otimes_B t^2)$. Then we compute that Ψ is left T-colinear:

$$(T \otimes \Psi) \circ (\rho_L \otimes M)(a \otimes m) = \sum_{i,k} t_k \otimes t_i \otimes \beta_i(\beta_k(a))m$$

on the one hand, and

$$(\Delta \otimes M) \circ \Psi(a \otimes m) = \sum_j t_j \otimes_R (\beta_j(t_i^1) \otimes_B t_i^2) \otimes_R \beta_i(a)m$$

on the other hand, equal elements of

$$T \otimes_R T \otimes_R M \xrightarrow{\cong} A \otimes_B A \otimes_B M, \ \ t \otimes u \otimes m \longmapsto t^1 \otimes_B t^2 u^1 \otimes_B u^2 m \tag{27}$$

since both map into the element $a \otimes_B 1_A \otimes_B m$.

Finally Ψ has inverse mapping defined by

$$\Psi^{-1}: T \otimes_R M \xrightarrow{\cong} A \otimes_B M, \ \ t \otimes_R m \longmapsto t^1 \otimes_B t^2 m \tag{28}$$

which follows from eq. (3) or [14, 2.2]. $\qquad\qquad\square$

Let $_BM$ be any B-module, $\mathcal{E} = \mathrm{End}\,_BM$ its endomorphism ring, $N = A\otimes_B M$ its induced A-module and $E = (\mathrm{End}\,_AN)^{\mathrm{op}}$ its endomorphism ring. We make note of the natural module N_E. The depth two structure on $A\,|\,B$ imparts on N an obvious left T-comodule structure with coaction Δ_N enjoying a *Hopf module* compatibility condition w.r.t. the left A-module structure, since A is a T^{op}-comodule algebra): $(a \in A, n \in N = A \otimes_B M)$

$$\Delta_N(an) = a_{(-1)}n_{(-1)} \otimes a_{(0)}n_{(0)} \in T \otimes_R N \qquad (29)$$

where the coaction on N is given by $n = a \otimes_B m \mapsto a_{(-1)} \otimes_R a_{(0)} \otimes_B m$.

Referring to our smash product decomposition above, we see that the proposition below is automatic if M is an A-module.

Proposition 4.4. *Let $_BM$ be a stable module. Then there is a left T^{op}-comodule algebra structure on E such that N is a Hopf module w.r.t. T and E and the algebra monomorphism $\mathcal{E} \hookrightarrow {}^{\mathrm{co\,T}}E$, $f \mapsto \mathrm{id}_A \otimes f$ is bijective.*

Proof. Define $\Delta_E : E \to T \otimes_R E$ via the canonical isomorphisms (using hom-tensor relation and T_R finite projective) where $h := \mathrm{Hom}\,(M, \Delta_N)$:

$$E \cong \mathrm{Hom}\,(_BM, {}_BN) \xrightarrow{\ h\ } \mathrm{Hom}\,(_BM, T \otimes_R N) \cong T \otimes_R E.$$

Denoting $\Delta_E(F) = F_{(-1)} \otimes F_{(0)}$, we note that

$$F_{(-1)} \otimes_R F_{(0)}(1_A \otimes_B m) = \Delta_N(F(1 \otimes m)) \qquad (30)$$

Then N is a (left-right) (E, T)-Hopf module since

$$\Delta_N(F(a \otimes m)) = \Delta_N(aF(1 \otimes m)) = a_{(-1)}F_{(-1)} \otimes_R F_{(0)}(a_{(0)} \otimes m).$$

It follows similarly that

$$\Delta_E(F \circ G) = G_{(-1)}F_{(-1)} \otimes_R F_{(0)} \circ G_{(0)}$$

and $\Delta_E(1_E) = 1_T \otimes 1_E$ since $\sum_i t_i\beta_i(1_A) = 1 \otimes_B 1 = 1_T$. Whence E is a left T-comodule algebra. The mapping $\mathcal{E} \to E$ is monic since A_B is faithfully flat. Clearly endomorphisms of the form $\mathrm{id}_A \otimes_B f$ for $f \in \mathcal{E}$ are coinvariants of Δ_E by eq. (30) and that Δ_N is $\rho_L \otimes \mathrm{id}_M$. For the converse, we first note that

$$M \cong {}^{\mathrm{co\,T}}(A \otimes_B M) \quad \text{via } m \mapsto 1 \otimes m \qquad (31)$$

since $T \otimes_R A$ is a Galois coring with coinvariants B (cf. [12] and [2, 28.19]). If $F_{(-1)} \otimes F_{(0)} = 1_T \otimes F$, it follows from the displayed mapping that $F = \mathrm{id}_A \otimes_B g$ for some $g \in \mathcal{E}$. $\square$

If T has an antipode satisfying a few axioms (e.g. [3]), one may moreover show that having a unitary and left T-colinear mapping $J : T \to E$ is equivalent to M being isomorphic to a direct summand of an A-stable module (cf. [24, 3.3]).

5. Hopf subalgebras and codepth two

Let C and D be two coalgebras over a field k. The author defined a notion of codepth two for a coalgebra homomorphism $C \to D$ [15], which is dual to the notion of depth two for an algebra homomorphism $B \to A$. In this section we recall the definition of codepth two and provide an example coming from Schneider's coGalois theory [23]. Let H be a finite dimensional Hopf algebra over k. A Hopf subalgebra K of H has coideal $K^+ = \ker \varepsilon_K$ and induces the coalgebra epimorphism $H \to H/HK^+$ which we observe to be codepth two in this section.

Let $g : C \to D$ be a homomorphism of coalgebras over a field k. Then C has an induced D-D-bicomodule structure given by left coaction

$$\rho^L : C \to D \otimes C, \ \rho^L(c) = c_{(-1)} \otimes c_{(0)} := g(c_{(1)}) \otimes c_{(2)},$$

and by right coaction

$$\rho^R : C \to C \otimes D, \ \rho^R(c) = c_{(0)} \otimes c_{(1)} := c_{(1)} \otimes g(c_{(2)}).$$

These two coactions commute by coassociativity; we denote the resulting D-D-bicomodule structure on C by $^D C^D$ later in this section. In a similar way, any C-comodule becomes a D-comodule via the homomorphism g, the functor of corestriction [2, 11.9]. Unadorned tensors between modules are over k, we use a generalized Sweedler notation, the identity is sometimes denoted by its object, and basic terminology such as coalgebra homomorphism, comodule or bicomodule is defined in the standard way such as in [2].

Recall that the cotensor product

$$C \, \square_D \, C = \{ c \otimes c' \in C \otimes C \,|\, c_{(1)} \otimes g(c_{(2)}) \otimes c' = c \otimes g(c'_{(1)}) \otimes c'_{(2)} \},$$

where we suppress a possible summation $c \otimes c' = \sum_i c_i \otimes c'_i$. For example, if $g = \varepsilon : C \to K$ the counit on C, $C \, \square_D \, C = C \otimes C$.

Recall that $C \square_D C$ is a natural C-C-bicomodule via the coproduct Δ on C applied as $\Delta \otimes C$ for the left coaction and $C \otimes \Delta$ for the right coaction [2, 11.3]. Then $\underline{\Delta} : C \to C \square_D C$ induced by Δ (where $\underline{\Delta}(c) := c_{(1)} \otimes c_{(2)}$) is a C-C-bicomodule monomorphism. As D-C-bicomodule it is split by $c \otimes c' \mapsto \varepsilon(c)c'$, and as a C-D-bicomodule $\underline{\Delta}$ is split by $c \otimes c' \mapsto c\varepsilon(c')$ for $c \otimes c' \in C \square_D C$. (Since $c_{(1)} \otimes g(c_{(2)}) \otimes c' = c \otimes g(c'_{(1)}) \otimes c'_{(2)}$, it follows that $g(c) \otimes c' = \varepsilon(c)g(c'_{(1)}) \otimes c'_{(2)}$, whence $c \otimes c' \mapsto \varepsilon(c)c'$ is left D-colinear.) For example, if $D = C$ and $g = \mathrm{id}_C$, then $C \square_C C \cong C$, since $\underline{\Delta}$ is surjective.

It follows that C is in general isomorphic to a direct summand of $C \square_D C$ as D-C-bicomodules: $C \square_D C \cong C \oplus *$. Left codepth two coalgebra homomorphisms have the special complementary property:

Definition 5.1. [15, 6.1] A coalgebra homomorphism $g : C \to D$ is *left codepth two (coD2)* if for some positive integer N, we have D-C-bicomodule isomorphism

$$C \square_D C \oplus * \cong C^N, \tag{32}$$

i.e., the cotensor product $C \square_D C$ is isomorphic to a direct summand of a finite direct sum of C with itself as D-C-bicomodules. Right codepth two coalgebra homomorphisms are similarly defined.

Let $D^* \to C^*$ be the algebra extension k-dual to $g : C \to D$. Various comodule structures also pass to modules over the dual algebras. Left coD2 quasibases are given for each $c \otimes c' \in C \square_D C$ by

$$c \otimes c' = \sum_{i=1}^{N} \eta_i(c \otimes c'_{(1)}) \alpha_i(c'_{(2)}) \otimes c'_{(3)} \tag{33}$$

where $\eta_i \in (C \square_D C)^{* \, D^*}$ and $\alpha_i \in \mathrm{End}\, ^D C^D$ are called *left coD2 quasibases* for the coalgebra homomorphism $g : C \to D$ [15]. The equation is analogous to the eq. (3). There is a right bialgebroid structure on $\mathrm{End}\, ^D C^D$ over the centralizer $C^{* \, D^*}$ [15].

Schneider introduces the following set-up in [23] for a Hopf algebra H with bijective antipode τ, which we call coGalois coextension because it is dual to Galois

H-extensions. Let C be a right H-module coalgebra. This means that in addition to being a coalgebra and right H-module, it satisfies the obvious compatibility conditions:

$$(ch)_{(1)} \otimes (ch)_{(2)} = c_{(1)}h_{(1)} \otimes c_{(2)}h_{(2)} \tag{34}$$

and $\varepsilon_C(ch) = \varepsilon_C(c)\varepsilon_H(h)$ for all $c \in C, h \in H$. Then there is the canonical coalgebra epi $p : C \to \overline{C} = C/CH^+$ where $H^+ = \ker \varepsilon_H$, the elements of vanishing counit, and CH^+ a coideal of C. We define $C \to \overline{C}$ to be *coGalois* in case the mapping

$$\text{can} : C \otimes H \longrightarrow C \,\square_{\overline{C}}\, C, \quad c \otimes h \longmapsto c_{(1)} \otimes c_{(2)}h \tag{35}$$

is bijective. Note that can does indeed have codomain in the cotensor product since $h - \varepsilon_H(h)1_H \in H^+$.

Proposition 5.2. *Suppose H is a finite dimensional Hopf algebra. If the coalgebra epimorphism $p : C \to \overline{C}$ defined above is coGalois, then it is left and right codepth two.*

Proof. It is not difficult to check that can is a left C-colinear and right $\overline{C}$-colinear homomorphism with respect to obvious C-$\overline{C}$-bicomodule structures on $C \otimes H$ and $C \,\square_{\overline{C}}\, C$ (via Δ_C and $(\text{id} \otimes p)\Delta_C$). Let $\dim H = n$. Then $C \,\square_{\overline{C}}\, C \cong C^n$ as C-$\overline{C}$-bicomodules, whence $C \to \overline{C}$ is right coD2.

Consider the variant coGalois mapping

$$\text{can}' : C \otimes H \longrightarrow C \,\square_{\overline{C}}\, C, \quad c \otimes h \longmapsto c_{(1)}h \otimes c_{(2)} \tag{36}$$

This is easily checked to be left $\overline{C}$-colinear and right C-colinear. But $\text{can}' = \text{can} \circ \Phi$ via the bijective mapping $\Phi : C \otimes H \to C \otimes H$ and its inverse defined by

$$\Phi(c \otimes h) = ch_{(1)} \otimes \tau(h_{(2)}), \quad \Phi^{-1}(c \otimes h) = ch_{(2)} \otimes \tau^{-1}(h_{(1)}) \tag{37}$$

Whence can' is bijective, so $C \,\square_{\overline{C}}\, C \cong C^n$ as $\overline{C}$-C-bicomodules, whence $C \to \overline{C}$ is left coD2. $\square$

Corollary 5.3. *Let H be a finite dimensional Hopf algebra and K a Hopf subalgebra of H. Then the canonical coalgebra epimorphism $p : H \to H/HK^+$ is codepth two.*

Proof. Follows from the proposition if we let $C = H$ be the obvious underlying right K-module coalgebra (where K takes the place of H in the proposition). We note that $\text{can} : H \otimes K \to H \,\square_{H/HK^+}\, H$ is split monic via the retract $H \otimes H \to H \otimes H$, $x \otimes y \mapsto x_{(1)} \otimes \tau(x_{(2)})y$, restricted to the image of can. Schneider [23, Theorem II] shows that the mapping can in eq. (35) is bijective if injective, and C is a projective right H-module. But H is free as a natural K-module by the Nichols-Zoeller theorem [21]. It follows that can is bijective, so that p is codepth two. $\square$

If K is normal in H, i.e., it is ad_ℓ-stable $a_{(1)}K\tau(a_{(2)}) \subseteq K$ and ad_r-stable $\tau(a_{(1)})Ka_{(2)} \subseteq K$ for all $a \in H$, then $HK^+ = K^+H$ [21, 3.4.4] so H/HK^+ is the quotient Hopf algebra $\overline{H}$.

Corollary 5.4. *Let H be any Hopf algebra with bijective antipode and K a normal finite dimensional Hopf subalgebra. Then the canonical epi $p : H \to \overline{H}$ is codepth two.*

Proof. This follows from the proposition and the proof of the previous corollary, except that we use a result of Schneider's [25] to conclude that H is free over K. $\square$

5.1. **Duality between codepth two and depth two.** Let k be a field. Suppose all k-algebras and k-coalgebras are finite dimensional in this subsection. In this case, there is a duality $M \mapsto M^*$ of finite dimensional C-D-bicomodules with finite dimensional A-B-bimodules, where $A = C^*$ is the dual algebra of C (with convolution multiplication) and $B = D^*$. The bimodule structure is given by $(a \cdot m^* \cdot b)(m) = a(m_{(-1)})m^*(m_{(0)})b(m_{(1)})$ in the obvious notation. We next show that a morphism of coalgebras $g : C \to D$ is codepth two if and only if its dual morphism $g^* : B \to A$ of algebras is depth two. Moreover, the bialgebroid of a codepth two extension defined in [15, 6.9] is anti-isomorphic to the bialgebroid S of the depth two dual algebra extension.

Let C and D be finite dimensional coalgebras, and $A = C^*$ and $B = D^*$ be their dual algebras. Of course, $g : C \to D$ is a coalgebra homomorphism if and only if $g^* : B \to A$, defined by $g^*(d^*) = d^* \circ g$ where $d^* \in D^* = \mathrm{Hom}_k(D, k)$, is an algebra homomorphism. Let $R = C_A(B)$, the centralizer of the B-B-bimodule induced on A by $g^* : B \to A$: thus $R = \{c^* \in A \,|\, \forall d^* \in B, g^*(d^*)c^* = c^*g^*(d^*)\}$.

Theorem 5.5. *Let $g : C \to D$ be a homomorphism of coalgebras. Then $g : C \to D$ is (left) coD2 if and only if the algebra homomorphism $g^* : B \to A$ is D2. Moreover, the R-bialgebroids $\mathrm{End}\,{}^D C^D$ and $\mathrm{End}\,{}_B A_B$ are anti-isomorphic.*

Proof. ($\Rightarrow$) We are given a split epimorphism of natural D-C-bicomodules $C^N \to C \,\square_D\, C$. Applying the duality mentioned above, we obtain a split monomorphism of natural D^*-C^*-bimodules $(C \,\square_D\, C)^* \to (C^*)^N$.

We note the [6, Lemma 3.5] which holds if C alone is finite dimensional: as A-A-bimodules, there is an isomorphism π,

$$A \otimes_B A \xrightarrow{\cong} (C \,\square_D\, C)^* \tag{38}$$

where $\pi(a \otimes c^*)(c \otimes d) = a(c)c^*(d)$ for $a, c^* \in A$, $c \otimes d \in C \,\square_D\, C$.

Then $A \otimes_B A \oplus * \cong A^N$ as B-A-bimodules. Whence $g^* : B \to A$ is left D2. Similarly, we argue that $C \to D$ is right coD2 $\Rightarrow g^* : B \to A$ is right D2.

($\Leftarrow$) We are given a split epi $A^N \to A \otimes_B A$ of B-A-bimodules. Note that $A^* = C$, so by dualizing we have a split monic $(A \otimes_B A)^* \to C^N$ of D-C-bicomodules. By eq. (38), $(A \otimes_B A)^* \cong C \,\square_D\, C$ as C-C-bicomodules. By corestriction then $C \,\square_D\, C \oplus * \cong C^N$ as D-C-bicomodules. Thus $g : C \to D$ is left coD2. Similarly, we argue that $g : C \to D$ is right coD2 if $g^* : B \to A$ is right D2.

The right bialgebroid $\mathrm{End}\,{}^D C^D$ over R described in [15, 6.9] is anti-isomorphic to the left bialgebroid $\mathrm{End}\,{}_B A_B$ described in section 3 via the mapping ($\alpha \in \mathrm{End}\,{}^D C^D, c^* \in A$)

$$\mathrm{End}\,{}^D C^D \to \mathrm{End}\,{}_B A_B, \quad \alpha \longmapsto \hat\alpha, \text{ where } \hat\alpha(c^*) = c^* \circ \alpha \tag{39}$$

We leave it as an exercise to show that this defines an anti-isomorphism of R-bialgebroids. For example, the transform of the target map on $r \in R$ is λ_r, since for each $c \in C$ $\widehat{t_R(r)}(\eta)(c) = r(c_{(1)})\eta(c_{(2)}) = \lambda_r(\eta)(c)$ by equation [15, (23)]. The counit ε_S of the transform of $\alpha \in \mathrm{End}\,{}^D C^D$ is evaluation at $1_{C^*} = \varepsilon_C$, $\varepsilon_S(\hat\alpha) = \hat\alpha(1_A) = \varepsilon_C \circ \alpha = \varepsilon_E(\alpha)$ by the counit equation [15, (25)]. Moreover, $\Delta_S(\hat\alpha) = \widehat{\alpha_{(1)}} \otimes \widehat{\alpha_{(2)}}$ since by equations [11, (7)], [15, (32)] ($\phi, \eta \in C^*, c \in C$)

$$(\hat\alpha_{(1)}(\phi) * \hat\alpha_{(2)}(\eta))(c) = \hat\alpha(\phi * \eta)(c) = (\phi * \eta)(\alpha(c)) = \phi(\alpha(c)_{(1)})\eta(\alpha(c)_{(2)})$$

$$= \phi(\alpha_{(1)}(c_{(1)}))\eta(\alpha_{(2)}(c_{(2)})) = (\widehat{\alpha_{(1)}}(\phi) * \widehat{\alpha_{(2)}}(\eta))(c)$$

where $*$ represents the convolution product on C^*. $\square$

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